of the elasticity of the fluid is, however, to accelerate the separation.

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Minimum Impulse Orbital Transfers

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SOLUTIONS have been obtained[1–3] for the absolute optimum two-impulse orbital transfer from a circular or apsidal terminal to an outer coplanar orbit, as well as from an inner orbit to an outer coplanar circular or apsidal terminal. The purpose of this paper is to extend these results so as to obtain the absolute optimum two-impulse transfer from an arbitrary inner terminal to an outer coplanar orbit and from an inner orbit to an arbitrary outer coplanar terminal. As in Refs. 2 and 3, terminal implies a radial distance and a velocity vector in an inverse square gravitational field.

Throughout this paper, the following idealizing assumptions have been made:

1) All given orbits are closed coplanar orbits having the same rotational sense.

2) The two orbits connected by the transfer orbit are such that they cannot intersect regardless of their relative orientation.

3) All velocity changes arise from instantaneous impulses.

4) The attracting body and the orbiting vehicle are constant point-masses in an idealized two-body system.

5) Transfers requiring more than two impulses are not considered.

The two orbits connected by the transfer orbit are called the inner and outer orbits, the inner orbit being that orbit with the smaller perigee altitude. Outer terminal and inner terminal refer, respectively, to terminals lying on the outer and inner orbits.

Of particular importance in the following derivations is a result of Ref. 4. There it was shown that, under the foregoing assumptions, the total impulse requirement ΔV_T for the absolute optimum two-impulse transfer from an arbitrary terminal whose radial distance is R_1 and whose velocity components are (u_0, v_0) to an arbitrary outer terminal whose radial distance is R_2 and whose velocity components are (u_F, v_F) is

$$\Delta V_T = [\{u_F + [2GMR_2/R_1(R_1 + R_2)]^{1/2}\}^2 + v_F^2]^{1/2} - [\{u_0 + [2GMR_1/R_2(R_1 + R_2)]^{1/2}\}^2 + v_0^2]^{1/2} \quad (1)$$

where G and M are, respectively, the gravitational constant and the mass of the attracting body.

Transfer from Inner Terminal to Outer Orbit

Because of the forementioned result of Ref. 4, the problem in this case becomes to vary the final terminal over the outer orbit, always making the optimum transfer [given by Eq. (1)] to the variable terminal, and to determine the location of the final terminal so that the outer orbit is achieved most economically.

It is mathematically convenient to express the total impulse requirement for the optimum terminal-to-terminal transfer in dimensionless multiples of circular velocity C_1 at the radial distance of the fixed initial terminal. Eq. (1) then becomes

$$\Delta V_T/C_1 = [\{\eta_F + [2/(1 + \rho)]^{1/2}\}^2 + \xi_F^2]^{1/2} - [\{\eta_0 + \rho[2/(1 + \rho)]^{1/2}\}^2 + \xi_0^2]^{1/2} \quad (2)$$

where

$$(u_i, v_i) = (C_1\eta_i, C_1\xi_i) \quad (3)$$

and

$$\rho = R_1/R_2 \quad (4)$$

If R_p and R_a are, respectively, the radial distances of the perigee and apogee of the orbit containing the outer terminal, the dimensionless perigee velocity η_p on the outer orbit is given by

$$\eta_p^2 = 2R_1R_a/R_p(R_a + R_p) \quad (5)$$

Since the rotational sense of the two orbits is the same, η_0, η_F, and η_p have the same sign, which is taken to be positive.

From the law of conservation of angular momentum,

$$\eta_F = \kappa\rho \quad (6)$$

where

$$\kappa = R_p\eta_p/R_1 = \text{const} \quad (7)$$

Combining the laws of conservation of angular momentum and total energy yields

$$\xi_F^2 = (\eta_p - 1/\kappa)^2 - (\kappa\rho - 1/\kappa)^2 \quad (8)$$

With Eqs. (6) and (8), $\Delta V_T/C_1$ is a continuous function of a single variable ρ in the closed interval

$$0 < R_1/R_a \leq \rho \leq R_1/R_p < 1 \quad (9)$$

Since neither term in Eq. (2) can vanish, $\Delta V_T/C_1$ has a continuous derivative in the interval of the variable ρ. Differentiating and simplifying gives

$$\frac{d(\Delta V_T/C_1)}{d\rho} = \frac{2^{-1/2}(2 + \rho)}{(1 + \rho)^{3/2}} \left\{ \frac{\kappa + \rho[2/(1 + \rho)]^{1/2}}{[\{\eta_F + [2/(1 + \rho)]^{1/2}\}^2 + \xi_F^2]^{1/2}} - \frac{\eta_0 + \rho[2/(1 + \rho)]^{1/2}}{[\{\eta_0 + \rho[2/(1 + \rho)]^{1/2}\}^2 + \xi_0^2]^{1/2}} \right\} \quad (10)$$

The second fraction inside the braces of Eq. (10) is always ≤ 1 so that $\Delta V_T/C_1$ has a positive first derivative when

$$\kappa + \rho[2/(1 + \rho)]^{1/2} > [\{\eta_F + [2/(1 + \rho)]^{1/2}\}^2 + \xi_F^2]^{1/2} \quad (11)$$

Assuming that Eq. (11) does not hold and using Eqs. (5–8) gives

$$1 \leq R_1/R_a \quad (12)$$

contradicting Eq. (9).

Thus, $\Delta V_T/C_1$ has a positive first derivative at every point in the interval, and the absolute optimum two-impulse transfer from the arbitrary inner terminal to the outer coplanar orbit is a transfer to the apogee of the outer orbit.

Using these results in Eq. (1) shows that the total impulse requirement for the optimum transfer is

$$(\Delta V_T)_{\min} = u_a + [2GMR_a/R_1(R_1 + R_a)]^{1/2} - [\{u_0 + [2GMR_1/R_a(R_1 + R_a)]^{1/2}\}^2 + v_0^2]^{1/2} \quad (13)$$

where u_a is the apogee velocity on the outer orbit.

From the results of Ref. 4 it can be shown that the apogee of the transfer orbit coincides with the apogee of the outer orbit, and that the apogee velocity u_2 on the optimum transfer orbit is

$$u_2 = C_1\sigma\left(\frac{2}{1 + \sigma}\right)^{1/2}\left[\frac{1 - \sigma(1 + B^2)^{1/2}}{(1 + B^2)^{1/2} - \sigma}\right] \quad (14)$$

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where

$$B = \xi_0/\{\eta_0 + \sigma[2/(1+\sigma)]^{1/2}\} \tag{15}$$

and

$$\sigma = R_1/R_a \tag{16}$$

This result, along with the conservation laws and the results of Ref. 4, is sufficient to determine the properties of the transfer orbit uniquely.

Transfer from Inner Orbit to Outer Terminal

The derivation of the solution in this case is exactly analogous to that of the preceding problem. In this case the initial terminal is allowed to vary over the inner orbit and the problem is to find the location of the variable terminal which minimizes the impulse requirement for the optimum terminal-to-terminal transfers.

Since now the initial terminal is variable, the total impulse requirement for the optimum terminal-to-terminal transfer is expressed in dimensionless multiples of circular velocity C_2 at the radial distance of the fixed outer terminal. Equation (1) then becomes

$$\Delta V_T/C_2 = [\{x_F + r[2/(1+r)]^{1/2}\}^2 + y_F^2]^{1/2} - [\{x_0 + [2/(1+r)]^{1/2}\}^2 + y_0^2]^{1/2} \tag{17}$$

where

$$(u_i, v_i) = (C_2x_i, C_2y_i) \tag{18}$$

and

$$r = 1/\rho = R_2/R_1 \tag{19}$$

Denoting the radial distances of the apogee and perigee of the inner orbit, respectively, by R_a' and R_p', the dimensionless perigee velocity x_p on that orbit is given by

$$x_p^2 = 2R_2R_a'/R_p'(R_a' + R_p') \tag{20}$$

As before, x_0, x_F, and x_p have the same sign, which is taken to be positive.

Corresponding to Eqs. (6–8) of the previous case, the conservation laws yield

$$x_0 = kr \tag{21}$$

and

$$y_0^2 = (x_p - 1/k)^2 - (kr - 1/k)^2 \tag{22}$$

where

$$k = R_p'x_p/R_2 = \text{const} \tag{23}$$

With Eqs. (21) and (22), $\Delta V_T/C_2$ is a continuous function of the single variable r in the closed interval

$$R_2/R_p' \geq r \geq R_2/R_a' > 1 \tag{24}$$

Since neither term in Eq. (17) can vanish, $\Delta V_T/C_2$ has a continuous first derivative in the interval of the variable r. Differentiating and simplifying gives

$$\frac{d(\Delta V_T/C_2)}{dr} = \frac{2^{-1/2}(2+r)}{(1+r)^{3/2}}\left\{\frac{x_F + r[2/(1+r)]^{1/2}}{[\{x_F + r[2/(1+r)]^{1/2}\}^2 + y_F^2]^{1/2}} - \frac{k + r[2/(1+r)]^{1/2}}{[\{x_0 + [2/(1+r)]^{1/2}\}^2 + y_0^2]^{1/2}}\right\} \tag{25}$$

The first fraction inside the braces of Eq. (25) is always ≤ 1 so that $\Delta V_T/C_2$ has a negative first derivative when

$$k + r[2/(1+r)]^{1/2} > [\{x_0 + [2/(1+r)]^{1/2}\}^2 + y_0^2]^{1/2} \tag{26}$$

Assuming that the inequality of Eq. (26) does not hold and using Eqs. (20–23) gives

$$1 \geq R_2/R_a' \tag{27}$$

contradicting Eq. (24). Thus $\Delta V_T/C_2$ has a negative first derivative throughout the interval, and the absolute optimum two-impulse transfer from the inner orbit to an arbitrary outer coplanar terminal is a transfer from the perigee of the inner orbit.

From Eq. (1) the total impulse requirement for the optimum transfer is

$$(\Delta V_T)_{\min} = [\{u_F + [2GMR_2/R_p'(R_p' + R_2)]^{1/2}\}^2 + v_F^2]^{1/2} - u_p - [2GMR_p'/R_2(R_p' + R_2)]^{1/2} \tag{28}$$

where u_p is the perigee velocity on the inner orbit.

From the results of Ref. 4 it can be shown that the perigee of the transfer orbit coincides with the perigee of the inner orbit, and that the perigee velocity u_1 on the transfer orbit is

$$u_1 = C_2 s\left(\frac{2}{1+s}\right)^{1/2}\left[\frac{1 - s(1+A^2)^{1/2}}{(1+A^2)^{1/2} - s}\right] \tag{29}$$

where

$$A = y_F/\{x_F + s[2/(1+s)]^{1/2}\} \tag{30}$$

and

$$s = R_2/R_p' \tag{31}$$

This result, combined with the conservation laws and the results of Ref. 4, is sufficient to determine the properties of the transfer orbit uniquely.

From the foregoing two results, it follows that the optimum two-impulse transfer between nonintersecting coplanar elliptical orbits is the Hohmann transfer from the inner perigee to the outer apogee, a result already established in Refs. 2 and 3 by a much longer method.

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Step-Temperature Effects on Direct Measurements of Drag

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It is well known that a local discontinuity in the surface temperature of a body moving through a fluid will markedly influence the heat transfer from the fluid to the body region over which the discontinuity exists. Much attention has been given to this step-temperature effect in heat transfer literature. The effect is especially evident in the measurement of heat transfer using plug-type calorimeters, where temperature mismatch is known to cause large errors.[1] The temperature history of the floating element of a direct-measuring skin-friction balance is similar to that of a plug

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